Multiple-scattering lidar from both sides of the clouds: Addressing internal structure

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Received 1 December 2007; revised 17 March 2008; accepted 19 May 2008; published 31 July 2008.

[1] Multiple-scattering (a.k.a. “off-beam”) lidar is an emerging technology in cloud remote sensing. It delivers, as in classic lidar ceilometry, cloud base altitude but also the cloud’s physical thickness $H$ as well as its optical depth $\tau$ (averaged over horizontal scales on the order of $H$). The value of $\tau$ in fact must lie beyond the range accessible by standard (i.e., single-scattering/on-beam) lidar profiling, namely, up to 3–4. A refined diffusion-theoretical model is presented here for signals from multiple-scattering lidar and applied, on the one hand, to retrieval algorithm development and, on the other hand, signal-to-noise ratio (SNR) estimation. SNRs are computed for LANL’s ground-based Wide-Angle Imaging Lidar (WAIL) system and NASA’s space-based Lidar-In-space Technology Experiment (LITE). The refinements are threefold and all about internal structure. First, the laser source is modeled as a collimated anisotropic exponentially distributed internal source rather than an isotropic point source at the cloud boundary; this opens the possibility of using $\delta$-Eddington rescaling to capture the forward peaked phase function more effectively within the diffusion framework. Second, stratification of the scattering coefficient is modeled as an increasing function of distance to cloud base; this strongly differentiates the signals when observed from above or from below. Finally, Cairns’ rescaling is applied to this conservative scattering problem to account for the systematic effects of random (turbulence-driven) internal variability at scales up to a few mean free paths.


1. Motivation, Context, and Outline

[2] It is commonplace to say that getting clouds right is an essential step in predictive climate science at both regional and global scales, for both near- and long-term forecasts. They are obvious elements in the radiation budget and hydrological cycle. They also participate actively in atmospheric aerosol processes, including their intricate chemistry in the aqueous phase as well as reactions on the surfaces offered by cloud particles [Ghan and Schwartz, 2007]. It is not as frequently voiced that clouds remain a significant challenge in remote sensing, and remote sensing is the only way we can assess them statistically with reasonable space-time sampling. As much as one would like to view clouds as known (or at least readily knowable) quantities, efforts with national and international reach such as DOE’s Atmospheric Radiation Measurement (ARM) program are predicated on the fact that we need to improve our knowledge and understanding of clouds. In recent years, it has become clear that possibly the strongest, and certainly the most uncertain, impacts of anthropogenic aerosol on the climate are mediated one way or another by clouds.


[3] Even though they are zeroth-order, questions about where we locate cloud boundaries, and the associated issue of cloud fraction, are already difficult. Part of this difficulty is that the answer depends inherently on the observational approach. And it should! Indeed, the spatial transition from clear to cloudy air is made fuzzy by nature herself through the complex interplay between advective and convective dynamics, thermodynamics, nucleation, turbulence, radiation, and so on, as any cloud modeler is well aware. It is therefore important to decide what instrumentation is best adapted to a given application that requires knowledge of cloud boundaries. This recommendation still stands when one asks the first-order question about what is going on inside the cloud boundaries in terms of instantaneous distributions of liquid and ice water content.

[4] Only after a cloud probing technique is selected, can one start meaningful discussions about precision, accuracy, robustness, reliability, sampling, etc. For instance, if climate modeling is the primary goal, then it is probably best to use remote sensing instrumentation that operates at wavelengths that matter most for the radiation budget; otherwise, a theory-based extrapolation across the EM spectrum is in order and this adds a vulnerability to the climate model.

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0148-0227/08/2007JD009666$09.00
Although there are many good reasons to invest large resources into radar and microwave technologies [e.g., Stephens, 1994], it is also important to keep working on passive and active techniques in the VIS through thermal IR region of the spectrum. Ultimately, we must recognize that, since different instruments “see” clouds differently, comprehensive observation of clouds for multiple purposes mandates multi-instrumental synergy and, if necessary, cloud data fusion.

[5] In this paper and its companion (A. B. Davis et al., Multiple-scattering lidar from both sides of the clouds: LITE from above, WAIL from below, manuscript in preparation, 2008, hereinafter referred to as Part 2), we revisit active remote sensing in the optical (VIS and Near-IR) spectrum, i.e., lidar, from a cloud perspective. The attending radiative transfer (RT) is dominated by scattering and the fundamental radiation transport physics will range from a ballistic/single-scattering regime to slow diffusion through extended regions where opacity is high, hence mean free paths (MFPs) are small. So far, atmospheric lidar has assumed the former situation. So much so that the famous “lidar equation” which predicts the lidar signal for a given atmospheric profile is an expression of two-way direct transmission and a single scattering through 180° (we assume here a so-called monostatic configuration where the transmitter and receiver are side by side, or even integrated as in micropulse lidars [Spinhirne, 1993]). In this case, the main constraint on the optical design is that the detector field of view (FOV) should contain all of the laser beam at all the ranges of interest; otherwise an “overlap function” must be determined and applied (at the cost of lost signal). Consequently, standard lidar FOVs are quite small, commensurate with the (typically diffraction-limited) divergence of the laser beam, on the order of a few mrad. For obvious reasons, this type of propagation will only cross the most tenuous clouds and penetrate only the first layers of their dense counterparts.

[6] We explore here the opposite asymptotic limit of RT where the lidar signal is dominated by beams energized by the multiple scattering source function. Once the cloud boundary nearest to the pulsed laser source is detected and ranged, this new signal modeling framework has no place for the ranging part of LIDAR (Light Detection and Ranging). However, it opens up new opportunities for cloud probing all the way to the nonilluminated boundary which may be very many MFPs away from the laser source. See schematic in Figure 1. In this scenario, the main constraint on the optical design is that the receiver FOV should contain as much as possible of the spatial pattern of reflected light excited by the laser beam. It has been shown theoretically [Marshak et al., 1995] and observationally [Davis et al., 1997] that, in reflection, the root-mean-square (RMS) radius of this radiative Green function is commensurate with the harmonic mean of the cloud thickness $H$, $\sim$km, and the “transport” MFP $t_o \sim$100 m. ($t_o$ and $H$ are the natural inner and outer scales of diffusion theory. Letting $g$ be the asymmetry factor and $\tau$ the cloud optical depth, their ratio is $(1 - g)\tau$ and we can moreover take $g \approx 0.85$ for all clouds of interest here.) We are thus contemplating an RMS radius of $\sim$0.3 km and a FOV that captures at least one full kilometer at cloud level; at typical ranges from ground, $\sim$1 km, this can translate to $\sim$1 radian, preferably even more.

[7] Curiously, multiple scattering in space-based lidar observations of clouds elicited strong interest [Flesia and Schwendimann, 1995; Miller and Stephens, 1999] before the same signal physics was investigated systematically for systems at much closer range to the cloud, either ground-based [Davis et al., 1999; Love et al., 2001a; Polonsky et al., 2005] or airborne [Cahalan et al., 2005]. A novel and interesting development is the idea that the multiple-scattering lidar system can be embedded in the cloud [Evans et al., 2003, 2006]. Originally, multiple scattering in lidar was generally viewed as a nuisance, and compensation methodology was developed to restore the utility of the single-scattering lidar equation. As far as we know, the first thrust in signal modeling and instrument development based on the notion that the multiple scattering component of the lidar signal can be useful was advanced by Bissonnette et al. [2002, and references therein] and described as “Multiple Field-Of-View” (MFOV) lidar; apart from standard ranging and quantifying aerosol density fluctuations, MFOV gives access to information on particle size distributions.

[8] To summarize, lidar is generally viewed as a mature technology addressing the atmospheric aerosol with unprecedented spatial and temporal detail, and now organized into growing networks in Europe, North and South America, and Asia that will in time be federated into a global network of networks. This growth and effort in standardization will lead to lidar data assimilation into NWP, regional and global transport studies, improved air quality forecasts, and so on. The new class of cloud-probing instruments supported by the present modeling study are bridging the gap between aerosol and clouds using the very same wavelengths and closely related detector physics. Conceivably, lidars with dense cloud capability will populate the same networks worldwide, capitalizing on the same investment in infrastructure; in some cases, the same transmitter may be used and only an extra detector will be added at a relatively low cost. In time, this kind of sensor development will help to bring on the scientific breakthrough we need to fully understand cloud-aerosol interaction. Active probing of both atmospheric components of the climate system with essentially the same instrumentation, the only difference being in the data collection and analysis, is a step in the right direction.

[9] The paper is dedicated to the forward modeling of the multiple-scattering lidar (MSL) signal using every resource radiative diffusion theory can bring to bear. In the following section, we review the rigorous time-dependent 3-D RT theory that supports MSL concepts in dense cloud remote sensing, up to the definition of space-time moments that play a key role in MSL signal phenomenology. In section 3, we establish a general diffusion-theoretical framework for multiple-scattering cloud lidar signal prediction. In section 4, we present diffusion results using a new and improved representation of the pulsed laser source, now as a sort of diffusely emitting source; for illustration, we apply it to a moment-based retrieval scheme. In section 5, we introduce two new cloud parameters (beyond $H$ and $\tau$) that describe internal cloud structure: one describes a macroscale gradient in opacity from bottom to top, and the other [adapted from Cairns et al., 2000] describes microscale random variability. In section 6, we apply this body of theory to the estimation of...
the raw MSL signal magnitude, and associated noises, for two radically different multiple-scattering cloud lidar systems. We summarize our results in section 7.

2. Preliminary Radiative Transfer and Probability Theory

2.1. A New Lidar Equation

A pulsed laser is for all practical purposes a physical instantiation of a Dirac $\delta$ source of illumination in time $t$, 3-D space $r$, propagation direction $\Omega$, as well as wavelength and possibly even polarization. By definition, the radiation field it excites is therefore a Green function $G(t, r, \Omega)$, the governing time-dependent 3-D RT equation being

$$
\left[c^{-1} \frac{\partial}{\partial t} + \Omega \cdot \nabla + \sigma(z)\right] G = \int_{4\pi} p(\Omega' \cdot \Omega) \cdot G(t, r, \Omega') d\Omega' + Q(t, r, \Omega),
$$

where we balance sinks on the left-hand side (LHS) and sources on the right-hand side (RHS) for the time-dependent radiance field in a small volume along beam direction $\Omega$. From left to right, we recognize the negative imprints of advection (total derivative $c^{-1}\frac{\partial}{\partial t} + \Omega \cdot \nabla$) and extinction, and positive counterparts for in-scattering and volume emission. Scattering and extinction coefficients are denoted $\sigma_s(z)$ and $\sigma(z)$ respectively and assumed to vary only in the vertical, but maintaining a constant ratio $\varpi_0$. The scattering phase function $p(\Omega' \cdot \Omega)$ is assumed axisymmetric (we will not concern ourselves here with cirrus) and spatially uniform. The complementary coefficient, $\sigma_a(z) = \sigma(z) - \sigma_s(z) = (1 - \varpi_0)\sigma(z)$, captures absorption processes as needed. As in most atmospheric RT texts, we denote $\Omega(\theta, \phi) = (\rho \cos \phi, \rho \sin \phi, \mu)^T$ in Cartesian coordinates using polar angles, where $\mu = \cos \theta$ and $\eta = \sqrt{1 - \mu^2}$.

We can assume no incoming radiance at the cloud boundaries and model the laser source internally as

$$
Q(t, r, \Omega) = \sum_\nu E_p \delta(x) \delta(y) \delta(t - z/c) \sigma_s(z) p(\mu) \cdot \exp \left(-\int_0^z \sigma(z') dz'\right), \quad 0 < z < H,
$$

Figure 1. Lidar observation of a dense cloud. Standard (single-scattering/on-beam) lidar is illustrated on the left, and multiple-scattering/off-beam lidar is illustrated on the right. We note the narrowness of the FOV in the standard case, as is required to restrict as much as possible the signal to a single backscatter, and the very wide FOV in the case of off-beam lidar, designed to capture all orders of scattering in the reflected laser light.
assuming vertical (z axis) beam alignment, a plane-parallel cloud of thickness $H$ at right angles, and total pulse energy $E_p$. Note from the writing of the third $\delta$ function that the instant $t = 0$ is when the laser pulse hits the cloud boundary at $z = 0$, precisely at $x = y = 0$. In this case, the direct beam is treated separately from the diffuse radiation field. Only the later is of interest in lidar on the detection side.

[12] Alternatively, we can set $G(t, \mathbf{r}, \mathbf{\Omega}) \equiv 0$ and model the highly directional laser source in the explicit statement of boundary conditions (BCs):

$$G(t, x, y, 0, \mathbf{\Omega}) = E_p \delta(t)(\delta(x)\delta(y)(\mathbf{\Omega} - \hat{z})), \mu > 0,$$

$$G(t, x, y, H, \mathbf{\Omega}) = 0, \mu < 0.$$  

(3)

where $\hat{z}$ orients the positive z axis. In this case, the resulting radiance field contains both direct and diffuse components. Apart from this interpretation of what is contained in $G(t, \mathbf{r}, \mathbf{\Omega})$ or not, the two ways of modeling the pulsed laser source are equivalent. Either way, we have now entirely determined the radiation transport in MSL.

[13] However, in this particular application, we are only interested in the reflected diffuse field at the illuminated boundary: $G(t, x, y, 0, \mathbf{\Omega})$, when $\mathbf{\Omega}_0 = \mu < 0$. More precisely, we assume an imaging detector is measuring this radiance at some finite standoff distance $d_{obs} > 0$ from the illuminated cloud boundary; that is, the MSL sensor is positioned at $\mathbf{r}_{obs} = (0, 0, -d_{obs})^T$. We denote the time-dependent cloud response at the detector $I(t_{\text{round-trip}}, \theta_{obs})$ where, by axial symmetry around the laser beam, we have no dependence on the azimuthal angle. From this vantage point, we subsample the axisymmetric boundary Green function for boundary illumination, $G(t, x, y, 0, \mathbf{\Omega}) \equiv G(t, \rho, 0, \mathbf{\Omega}_{obs}(\rho))$ where

$$t = t_{\text{round-trip}} - \frac{1}{c} (1 + \cos \theta_{obs})d_{obs},$$

$$\rho(\theta_{obs}) = \sqrt{x^2 + y^2} = d_{obs} \tan \theta_{obs},$$

hence, $\theta_{obs}(\rho) = \tan^{-1}(\rho/d_{obs})$, and

$$\mathbf{\Omega}_{obs}(\rho) = \left(-\sin \theta_{obs}(\rho), 0, -\cos \theta_{obs}(\rho)\right)^T.$$  

(4)

We assume here, for simplicity, that radiation is sampled in the $y = 0$ half-plane with $\rho > 0$. By deriving $I(t_{\text{round-trip}}, \theta_{obs}(\rho))$ from $G(t, \rho, 0, \mathbf{\Omega}_{obs}(\rho))$ and $d_{obs} < \infty$ using (1) and (4), with either (2) or (3) to describe the source, we have completely specified the new lidar equation for multiple-scattering systems in the framework of RT theory.

[14] To illustrate this point with standard/on-beam lidar, we compute only the single-scattering term; using (2) and the well-known propagation kernel for (1), we have

$$I_1(t + 2d_{obs}/c) = \int_{-\infty}^{+\infty} \int_{-\infty}^{H} Q(t, x, y, z, -\hat{z}) \exp\left(-\int_0^z \sigma(\zeta)d\zeta\right) \frac{dxdydz}{(z + d_{obs})^2},$$

$$E_p c\sigma(\zeta)p(-1) \bigg|_{z=d_{obs}}.$$  

(5)

for a uniform medium. The exponential term decays very rapidly over a few MFPs (recalling that 1 MFP = $1/\sigma$), and therein is the limitation of penetration by standard lidar into dense clouds. In MSL, by contrast, we are interested in the full solution of the 3-D RT equation.

[15] In the limit $d_{obs} \to \infty$, a reasonable approximation for an orbital detector, the connections in (4) still make sense by taking the simultaneous limit $\theta_{obs} \to 0$, keeping $\rho$ constant. We thus denote the detector response as $R(t, \rho)$, after accounting for the large but finite time delay, and the last connection simplifies to $\mathbf{\Omega}_{obs}(\rho) \equiv -\hat{z}$.

2.2. Spatial and Temporal Moments

[16] To summarize the above, we need to obtain from theory, computation, remote lidar measurements, or some combination of the above, the time-dependent axisymmetric (equivalent) reflectance field $R_{obs}(t, \rho) = \pi G(t, \rho, 0, \mathbf{\Omega}_{obs}(\rho))/E_p$, which is normalized by the pulse energy. Temporarily ignoring angular sampling and truncation issues in real measurements, we define

$$R(t, \rho) = \frac{2\pi}{E_p} \int_0^{\pi/2} \frac{\cos \theta |G(t, \rho, 0, \mathbf{\Omega}(\theta, 0))| \sin \theta d\theta}{},$$

(6)

as the local time-dependent reflected flux field.

[17] Largely to improve the signal-to-noise ratio (see section 6), it is of interest to use spatial and/or temporal integrals of the observed $R(t, \rho)$. We are particularly interested in its statistical moments when it is viewed as a probability density function (PDF) for escape in reflection. We will therefore estimate:

$$R = \frac{2\pi}{E_p} \int_0^{\infty} \frac{d\tau}{d\rho} \int_0^{\infty} R(t, \rho) d\rho d\tau,$$

(7)

the cloud’s albedo (for steady, uniform and normal illumination), and moments

$$\langle t^q \rangle = \frac{2\pi}{E_p} \int_0^{\infty} t^q d\tau \int_0^{\infty} R(t, \rho) d\rho d\tau (q = 1, 2, \ldots),$$

(8)

$$\langle \rho^2 \rangle = \frac{2\pi}{E_p} \int_0^{\infty} \frac{d\rho}{d\tau} \int_0^{\infty} \rho^2 R(t, \rho) d\rho d\tau.$$  

(9)

Angular brackets will always denote an average over space and/or time.

[18] Note that the moment estimations in (9)–(8) are immune to uncertainties in a multiplicative constant for $R(t, \rho)$. From an observational standpoint, and in sharp contrast with the estimation of cloud albedo in (7), absolute calibration is not required. But the easier task of flat-fielding of the imager’s focal plane array is necessary.

[19] Of course, real-world MSL observations give us no information on $G(t, \rho, 0, \mathbf{\Omega}_{obs}(\rho))$, hence on $R(t, \rho)$, outside of the receiver’s FOV (i.e., the actual upper limit of all the
above integrals over $\rho$ is finite). Moreover, for each value of $\rho$ we only get one value of $\theta$ in (4). The latter problem is resolved by using an angular model to convert an observed radiance into a boundary flux. The former problem is best addressed by designing MSL instruments with the widest possible FOV, such that it contains at least a couple of the Green function’s e-folding distances away from the axis; we can then assume that the residual truncation in both numerators and denominators in (9)–(8) does not bias the estimates. We return to these two observational issues respectively in sections 3.3 and 6.1.

2.3. Moment Estimation in Fourier-Laplace Space

[20] Moment integrals in (8)–(9) are easy to compute by manipulation of transforms in Fourier-Laplace space. In probability theory, the Fourier or Laplace transforms of a PDF is called its “characteristic” or “moment-generating” function. Which transform is used depends on the support of the PDF. In our application, we need both Laplace for time $t \in [0, \infty)$ and 2-D Fourier for position $\vec{r} = (x, y) \in \mathbb{R}^2$ in the $z = 0$ plane.

[21] We are thus interested in

$$R(s, k) = \int_0^\infty \int_{-\infty}^{\infty} \exp\left(-st + ik \cdot \vec{r}\right) G(t, \vec{r}, z) \, dx \, dy$$

$$= R \times \langle \exp(-st + ik \cdot \vec{r}) \rangle$$

(10)

for the time-dependent 2-D reflectance field. It is not hard to see that coefficients of Taylor expansions of $R(s, k)$ at $s = 0$ and $k = 0$ can be used to compute spatial and temporal moments. By translational and rotational symmetries that carry over from physical to Fourier space, we have $R(s, k) \equiv \tilde{R}(s, k)$ and, specifically, we need to compute albedo $R = \tilde{R}(0, 0)$, as well as moments

$$\langle R^q \rangle = \frac{1}{R} \left( \frac{\partial}{\partial s} \right)^q \tilde{R} \bigg|_{s=0, k=0}, \quad (q = 1, 2, \ldots),$$

(11)

$$\langle \rho^2 \rangle = -\frac{2}{R} \frac{\partial^2 \tilde{R}}{\partial k^2} \bigg|_{s=0, k=0}. \quad (12)$$

3. A Diffusion-Based Framework for MSL Signal Prediction

[22] We still need a physically reasonable theory for $R(t, x, y) \equiv \tilde{R}(t, \rho)$ or, equivalently, $\tilde{R}(s, k)$ in order to use the above definitions and relations that predict the multiple-scattering cloud lidar signal and derived moments.

3.1. Simplified Transport Equations

[23] Now consider dense clouds, say, through which one cannot detect the silhouette and maybe not even the general direction of the sun in the transmitted radiance field. According to Bohren et al. [1995], this means optical thickness $\tau$ in excess of $8-10$. We can safely assume that such light is transported via diffusion, the well-known approximation to RT per se. In other words, all is as if photons detected in transmission or reflection were particles executing typically long convoluted random walks starting at the localized and collimated source and ending at a cloud boundary.

[24] A classic approach to diffusion theory (a.k.a. “P,”) is to truncate the spherical harmonic expansion of the Green function radiance field at order one:

$$G(t, r, \Omega) \approx \frac{|J(t, r) + 3 \vec{F} \cdot \vec{F}(t, r)|}{4\pi}$$

(13)

where we denote the zeroth- and first-order angular moments as

$$J(t, r) = \int_{4\pi} G(t, r, \Omega) d\Omega, \quad \vec{F}(t, r) = \int \Omega G(t, r, \Omega) d\Omega, \quad (14)$$

respectively, the scalar (a.k.a. actinic) flux and vector flux. Accordingly, one assumes

$$p(\Omega \cdot \Omega') \approx [1 + 3 g \Omega \cdot \Omega'] / 4\pi$$

(15)

for the phase function where $g$ is the asymmetry factor (mean value of $\Omega \cdot \Omega'$). We note from the onset that (15) is a poor representation of the phase function of cloud droplets, most notably, the forward diffraction-induced peak is absent. By the same token, (13) is a poor representation of radiance anywhere near the highly collimated laser beam. We will treat these obvious problems separately further on, and thus improve the accuracy of the diffusion model in MSL.

[25] After substitution of (13) and (15) into (1), equations for this simplified transport theory are derived by angular integration term by term over $4\pi$, once directly, and once after multiplication by $\Omega$ [Case and Zweifel, 1967]:

$$\frac{1}{c} \frac{\partial J}{\partial t} + \nabla \cdot \vec{F} = -\sigma_s(z) J + q_s(t, r);$$

(16)

$$\frac{1}{c} \frac{\partial \vec{F}}{\partial t} + \nabla J / 3 = -\sigma_s(z) \vec{F} + \phi_e(t, r).$$

(17)

In (17), a new and important coefficient appears: “transport” extinction,

$$\sigma_t(z) = (1-g) \sigma_s(z) + \sigma_d(z) = (1-\varpi_0) \sigma(z),$$

(18)

where $\varpi_0 = \sigma_d(z)/\sigma(z)$ is the previously introduced single-scattering albedo (assumed constant here). The transport MFP $\ell_t$ mentioned in the introductory section is given in terms of local variables by $1/\sigma_t(z)$. As in (14), we define

$$q_s(t, r) = \int_{4\pi} Q(t, r, \Omega) d\Omega, \quad \phi_e(t, r) = \int_{4\pi} \Omega Q(t, r, \Omega) d\Omega. \quad (19)$$

[26] The “continuity” equation for radiant energy (16) is exact. Its counterpart for momentum in (17) is where the effect of the order-one truncation is felt, and it is further simplified by neglecting the time derivative. We thus obtain the “constitutive” equation:

$$\vec{F} = -\frac{1}{3 \sigma_t(z)} \nabla J + \phi_e(t, r) / \sigma_t(z).$$

(20)
and the homogeneous BC at \( z = H \) is unchanged. Note that, since flux alone tells us nothing about directionality, we are now effectively modeling the source as point-wise and pulsed but isotropic in the \( \mu > 0 \) hemisphere.

Equations (24) and (25) express the least usual (third) type of BCs that occur in generic applications of diffusion-type PDE problems, both time-dependent (parabolic) or steady state (elliptical): they involve the density \( J \) at the boundary and the boundary crossing current \( F_z \), equivalently, \( J \) and its normal derivative of \( J \) from (20). BCs can thus be expressed as a variable mixture of Dirichlet/first-type (fix \( J \)) and Neumann/second-type (fix \( F_z \)) BCs:

\[
J(t, \bar{\rho}, 0) + 3\chi F_z(t, \bar{\rho}, 0) = 4q_0(t, \bar{\rho})
\]

\[
J(t, \bar{\rho}, 0) - 3\chi F_z(t, \bar{\rho}, H) = 0.
\]

Although often referred to as “mixed” BCs, these are known technically as “Robin” BCs [Eriksson et al., 1996]. At any rate, they are the most general BCs we will need to consider in MSL signal modeling.

When \( q_0(t, \bar{\rho}) \) does not vanish, the BC mixing factor \( \chi \) can differ from its 2/3 value in (25), but typically not very much (at least in the most common transport applications). This is basically a tuning parameter that was introduced by early neutron transport theorists [e.g., Davison, 1958] to help diffusion theory reproduce high-precision solutions of the transport equations in critical applications; this boost in accuracy is naturally applied where diffusion is at its weakest, namely, boundaries. The physical interpretation of \( \chi \) is that of an “extrapolation length” measured in transport MFPs. Indeed, in the absence of anisotropic internal sources, (20) tells us that \( F_z(t, \bar{\rho}, 0) = -[\partial_z J / 3\sigma_\perp(t)]_{z=0} \), and similarly at \( z = H \). By substitution into (27), the LHS reads as a linear extrapolation formula for \( J \), given its derivative along the \( z \) axis, over a distance \( \sigma_\perp(t) \) into the \( z < 0 \) region; we have a similar reading of the BC at \( z = H \) going into the \( z > H \) region.

### 3.3. Fields Observable With MSL

The quantity of prime interest in MSL is local/ instantaneous reflectivity, i.e., the outgoing flux normalized by total energy:

\[
R(t, x, y) = \frac{F_+(t, x, y, 0)}{E_{\text{tot}}} = \frac{J(t, x, y, 0)/2 - F_z(t, x, y, 0)}{2E_{\text{tot}}}
\]

where \( E_{\text{tot}} \) is the space-time integral of \( q_\perp(t, \mathbf{r}) \). Invoking the BC at \( z = 0 \) in (24), we can express this basic cloud response simply as

\[
R(t, x, y) = \frac{J(t, x, y, 0)}{2E_{\text{tot}}}
\]

If the isotropic boundary source model in (27) is used for the BCs, then \( J \) and \( F \) necessarily contain the incident flux. We must therefore compute the required space-time reflectivity field in (28) from

\[
R(t, x, y) = \frac{F_+(t, x, y, 0)}{E_{\text{tot}}} = \frac{J(t, x, y, 0) - 2q_0(t, x, y)}{2E_{\text{tot}}}
\]
where $E_{\text{tot}}$ is defined as in (28) but for $q_0(t, x, y)$, hence without integrating over $z$.

[34] Finally, we recall that at cloud boundaries (and elsewhere) diffusion theory only predicts flux. A zeroth-order estimate of cloud-leaving radiance is given by $R(t, x, y)/\tau$, a Lambertian assumption which is not unreasonable for highly scattered light. A first-order angular model will use (13). This radiance-to-flux conversion can be done with better angular models, and should be for actual cloud remote sensing applications [cf. Polonsky et al., 2005].

[35] Determination of the new lidar equation within the diffusion approximation, as formulated in the space-time domain, is now complete. Moreover, several options are available to control its degree of fidelity in source representation.

3.4. Formulation in Fourier-Laplace Space

[36] Fourier-Laplace transformation of the PDE system in (16) and (20), with BCs (24) or (25), leads to a class of analytically tractable problems for our representations of pulsed laser sources in the case of either constant coefficients, or simple enough variability models.

[37] Letting $r = (x, y, z)^T = (\tilde{p}, z)^T$, we define

$$J(s, \tilde{p}, z) = \int_0^{\infty} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \exp(-st + i\tilde{k} \cdot \tilde{p}) J(t, \tilde{p}, z) \, dt \, \tilde{p} \, dz.$$  

(31)

We similarly transform all the components of $F(t, \tilde{p}, z)$, yielding $\tilde{F}(s, \tilde{p}, z)$. We can now think of $(s, \tilde{p})$ as parameters rather than independent variables, hence the “;” separator.

[38] Furthermore, let $F = (\tilde{F}_h, \tilde{F}_e)^T$, similarly for $q_E$ and we recall that $\nabla = (\partial/\partial \tilde{p}, \partial/\partial z)^T$ transforms to $(i\tilde{k}, d/dz)^T$. Because of the axial symmetry of the source ($\tilde{F}_h \equiv 0$, the (vector + scalar) PDE system in (20)–(21) reduces to two coupled 1D ODEs:

$$\tilde{J} / 3 = -\sigma_a(s, k) \tilde{F}_z + \tilde{q}_E,$$

(32)

$$\tilde{F}_z = \left( \frac{s}{c} + \frac{k^2}{3\sigma_a(z)} \right) \tilde{J} + \tilde{q}_j.$$  

(33)

The latter ODE is and expression of energy conservation (with transport) along the $z$ axis where local time variation and horizontal divergence of $J$ are recast as “effective” absorption processes:

$$\sigma_a^{(e)}(s, k, z) = s/c + k^2/3\sigma_a(z).$$  

(34)

This is a coefficient that, in general, is stratified differently than $\sigma_a(z)$ ($x = s, a, t$), which all vary together ($\sigma_0$ and $g$ being assumed constant).

[39] From (22), Fourier-Laplace transformed internal source terms used in MSL are

$$\tilde{q}_j(s, k) = E_p \sigma_s(z) e^{-(s/c)z} \int_0^{z} \sigma(z') dz', \quad \tilde{q}_E(s, k) = g \times \tilde{q}_j(s, k),$$

independent of $k$.

[40] The general BCs in (27) become

$$\tilde{J}(s, k; 0) + 3\tilde{q}_F(s, k; 0) = 4\tilde{q}_0(s, k),$$

$$\tilde{J}(s, k; 0) - 3\tilde{q}_F(s, k; H) = 0,$$

(36)

(37)

where $\tilde{q}_0(s, k) \equiv 0$ and $\chi = 2/3$ if the distributed internal source model in (35) is used. If the boundary point source model is used, $\tilde{q}_0(t, x, y)$ in (26) leads to $\tilde{q}_0(s, k) \equiv \tilde{E}_p$ in (36).

[41] We recall finally that, in multiple-scattering cloud lidar, our interest is limited to $R(s, k) = \tilde{J}(s, k; 0)/2E_p$, or, when using the boundary source option, $[\tilde{J}(s, k; 0) - 2\tilde{q}_0(s, k)]/2E_p$.

4. Laser Source as a Collimated Beam Decaying Exponentially Inside the Cloud

4.1. Forward Model for MSL Observables

[42] In this first application of the general diffusion framework, we start with the same assumptions as Davis et al. [1999] in their proof-of-concept paper on MSL observation of dense clouds: conservative scattering ($\sigma_a = 0$) and uniform cloud (constant $\sigma_s = \sigma_a$ and $\sigma_t$). However, rather than BCs with a source term, we use the more accurate representation of the pulsed laser beam formalized in (22), hence (35), as an exponential distribution of anisotropic internal sources. Assuming a unit pulse $(E_p = 1)$, we must therefore solve

$$\tilde{F}_z = -\sigma_a(s, k) \tilde{J} + \sigma_a e^{-(\sigma_a/\sigma_t)z},$$

$$\tilde{J}/3 = -(1-g)\sigma_F + g\sigma_a e^{-(\sigma_a/\sigma_t)z}.\phantom{\text{(38)}}$$

(38)

Accordingly, we take $\tilde{q}_0(s, k) = 0$ in the general BCs (36)–(37) and set $\chi = 2/3$, leading to the Fourier-Laplace version of (24):

$$\tilde{J} + 2\tilde{F}_z|_{z=0} = 0, \quad \tilde{J} - 2\tilde{F}_z|_{z=H} = 0.$$  

(39)

We anticipate dependence on both $\tau$ and on $g$, not just on $(1-g)\tau$. Also notice that $s$ enters the exponential source term, as an effective $\sigma_s$, but $k^2/3\sigma_t$ does not. This transformed 3-D time-dependent problem is not formally identical to a solar two-stream problem, at least when $k \neq 0$.

[43] Following the steps described in the previous sections, we find at zeroth-order

$$R = 1 - T, \quad T = \frac{5 - e^{-\tau}}{3(1-g)\tau + 4},$$

(40)

i.e., the well-known expressions for cloud albedo $R$ and transmittance $T$ for normal solar illumination and no absorption [Meador and Weaver, 1980]. Higher-order terms in $k$ and $s$ yield moments such as

$$\langle \rho_2 \rangle / \langle H \rangle^2 = \frac{20}{9} \frac{1}{(1-g)\tau} \times [1 + C_{\rho,2}(\tau, g)],$$

$$\langle \rho t \rangle / \langle H \rangle = \frac{5}{3} \times [1 + C_{\rho t,1}(\tau, g)],$$

$$\langle (ct) \rangle / \langle H \rangle^2 = \frac{2}{3} (1-g)\tau \times [1 + C_{ct,2}(\tau, g)],$$

$$\langle (ct)^3 \rangle / \langle H \rangle^3 = \frac{4}{7} (1-g)\tau^2 \times [1 + C_{ct,3}(\tau, g)].$$

(41)
Note that the temporal statistics are now expressed as moments of path \( ct \) in units of \([\text{length}]^3\). In the above relations, we also normalized all the moments with \( H^q(q = 1, 2, 3, \text{as needed})\). Dependencies on cloud parameters on the RHS can thus be expressed only with dimensionless quantities: optical depth \( \tau = \sigma H \) and \( g \). In all cases, we give explicitly the dominant term for large \( \tau \) and express the remainder as a multiplicative correction term that goes to unity (the Cs vanish) as \( \tau \to \infty \). This representation emphasizes the fact that ratios of different moments (once expressed in the same physical units) are not constants, a remarkable property that is not found for transmitted light [Davis and Marshak, 2002]. This feature is of vital importance in MSL-based remote sensing and we can trace it to the characteristically balanced mixture of low and high orders of scattering in reflected light.

It suffices here to say that the preasymptotic corrections \( C_{\omega q}(\tau, g) \) have the form of rational functions of \( \tau \) and \( g \) (also containing rapidly decaying terms in \( e^{-}\)) that become increasingly complex as \( g \) increases. They all decay slowly in \( 1/\tau \) as \( \tau \to \infty \). Complete expressions for the above moments are supplied as auxiliary material\(^1\) in the form of FORTRAN 77 code, and further details on their derivation are provided by A. B. Davis et al. (Space-time Green functions for diffusive radiation transport, in application to active and passive cloud probing, submitted to Light Scattering Reviews, 2008).

As previously mentioned, a weakness of diffusion-based radiation transport modeling is the smooth one-parameter phase function in (15) whereas real-world phase functions have prominent forward peaks [Deirmendjian, 1969]. We can partially mitigate this disconnect by applying the classic \( \delta \)-Eddington rescaling [Joseph et al., 1976]. The phase function is recast as a combination of a \( \delta \) function in the forward direction (physically, just prolonged ballistic propagation) and a complementary term with two spherical harmonics. In the absence of absorption, this results in a rescaling given by

\[
\sigma'(z) = (1-f)\sigma(z), \quad (1-g') = (1-g)/(1-f),
\]

where \( f \) is the fraction of "\( \delta \) scattering." This operation decreases \( \sigma \equiv \sigma_s \) (increases the MFP), but leaves \( \sigma_t \) invariant in (18). It therefore has no effect on Davis et al.’s [1999] model since it depends only on \( \tau_t \equiv (1-g)\tau = \sigma_t H \).

A popular choice is \( f = g^3 \) because it fits the spherical harmonic coefficients of the Heney and Greenstein [1941] model phase function up to order 2, hence

\[
g'(g-f)/(1-f) = g/(1+g).
\]

For liquid clouds, where \( g \approx 0.85 \), we get \( \sigma' \approx 0.28\sigma \) and \( g' \approx 0.46 \).

Figure 2 shows the dependence of the four normalized responses from (41) as functions of \( (1-g)\tau = (1-g')\tau' \) when \( g = 0.85 \) and \( g' = 0.46 \) in log-log axes, using an RMS format for the second-order moments in space and time, and a 1/3 power for the third-order moment in path. Figure 2 also demonstrates the validity of the updated diffusion model for its intended purpose (i.e., \( (1-g)\tau \geq 1 \)), and especially when using the rescaling based on \( g' \) in (43). This validation is based on a comparison of the diffusion-theoretical predictions in (41) with MC solutions of the more general RT problem in (1)-(2). We note that the preasymptotic corrections in (41) are clearly important, especially for the higher-order moments of path \( ct \) and/or when the \( g' \) rescaling is not used. Finally, the importance of using \( g' \) rescaling in diffusion theory is underscored by the violations of basic statistical inequalities (e.g., mean \( > \) RMS) when \( g = 0.85 \) and \( \tau \leq 10 \).

### 4.2. Cloud Property Retrievals

We note that cloud albedo and transmittance in (40) are available in lidar studies not only as the space/time integral of the MSL signal, but also from the solar or lunar background “signal” of well-calibrated lidar systems of any ilk, at least during daytime. After conversion from radiance to flux (or, better still, working with 1-D steady state radiance models) and accounting for the slant incidence

\(^1\)Auxiliary materials are available in the HTML. doi:10.1029/2007JD009666.
angle, they can be used to retrieve \( \tau \). However, space-based and airborne instruments depending on \( R \) [Platt et al., 1998; Yang et al., 2008] will not be as sensitive as their ground-based counterparts that use \( T \) [Chiu et al., 2007] when \( \tau \) is large since \( \partial \ln R / \partial \ln \tau \to 0 \) while \( \partial \ln T / \partial \ln \tau \to 1 \) as \( \tau \to \infty \).

[48] Only higher-order MSL observables however give us access to \( H \) and, furthermore, they can also deliver an estimate of \( \tau \), all of this without need for absolute calibration. From the cloud remote sensing perspective, we are glad to see that, starting with the asymptotic trends in (41), the four responses vary differently with \( \tau \). By obtaining from MSL observations any two of the four moments, we can therefore infer the two targeted cloud properties, namely, \( \tau \) (given \( g \)) and then \( H \).

[50] In Figure 3, we demonstrate the basic principle using the ratio \( <(r^2)>(ct)^2 \), which is sensitive to scaled optical depth \( (1-g)\tau \) but apparently almost completely insensitive to the specific choice of \( g \) itself (i.e., we can use either \( g \) or \( g' \) curves). Knowing \( \tau \) (since we can safely prescribe \( g \), hence \( g' \)), any one of the lidar moments will give us \( H \) by comparing the prediction in (41) and its observed counterpart. In Figure 3, we propose to use \( H/(ct) \) which has the desirable property of being quite insensitive to \( \tau \), especially if we use \( g' \) rather than \( g \). In this MSL cloud remote sensing demo, we have propagated graphically an assumed uncertainty of \( \pm 17\% \) on the second-order moment ratio, leading to \( \pm 13\% \) on \( \tau \) and less than \( \pm 1\% \) on the factor that converts \( (ct) \) to \( H \).

[51] We have described here only moment-based retrieval methods in MSL observation of clouds. However, it is not always possible to estimate accurately the required moments from MSL data, say, due to an insufficiently large FOV resulting in a truncation of the observed Green function’s tail. In Part 2, we revisit the direct PDF-based methodology of Polonsky et al. [2005] that overcomes this looming instrumental problem.

[52] What is the spatial resolution of an MSL-inferred cloud property? And what is the optimal spatial sampling? The answers of such questions usually involve the exposure time while the prevailing wind advects the cloud above a ground-based sensor, the transmitted beam divergence and platform velocity in airborne or satellite observations. It is remarkable that here the answers depend more on the cloud being observed. Indeed, by its very nature, the MSL signal originates from the whole volume of the cloud as defined by a horizontal area, say, a couple of times larger than \( \tau (\sqrt{r^2}) \ldots \) which varies from cloud to cloud. We recall that the radius of that circular area is known as the “radiative smoothing scale” [Marshak et al., 1995], and it defines the minimum pixel size at which passive cloud remote sensing can be performed without too much risk of contamination by adjacency effects. In MSL as well, any cloud structure smaller than this is smoothed by the radiative diffusion process. This smoothing scale also defines the minimum sampling distance (or time interval) that one would want to use in operational MSL observations. Anything faster would mean overlap in the radiative Green functions being measured, hence redundant cloud information. Anything slower will combine into a single observation the Green functions of cloud sectors that may have different physical properties; we are then faced with a nonlinear subresolution variability problem.

[53] If we absolutely had to set a specific value for the MSL “footprint” and the sampling scale, we would look at Figure 3 and note that \( \sqrt{\langle r^2 \rangle} \) is on the order of \( H \), which is typically \( O(1) \) km and, generally speaking, is the least variable of the cloud parameters (at least within a given cloudy layer); at the same time, its dependence on \( \tau \) is relatively weak (a fluctuation over an order of magnitude only yields a factor of \( \approx 3 \)). At any rate, MSL’s inherent resolution is ideally suited for RT studies since smaller fluctuations affect only the bulk transport and call for a stochastic model (see below) while larger ones excite “adjacency” effects that call for a deterministic 3-D RT approach.

5. Impact of Internal Cloud Structure on MSL Observables

5.1. Stratification

[54] Stratiform clouds are expected to exhibit internal stratification. For instance, in their “convective cores,” liquid water content is predicted and widely observed...
[Pawlowska et al., 2000] to follow the adiabatic gradient, a
linear trend in \( z \) over the vertical extent of the cloud. This
classic result from the baseline parcel theory in cloud
microphysics (number density assumed constant) leads to a
2/3 power law in extinction from straightforward dimen-
sional analysis. Formally, and depending on what side of the
cloud is being illuminated by the laser source, we can write
this as

\[
\sigma_{\ell}(\gamma; z) = \sigma (1 + \gamma) (z/H)^{\gamma}, \text{or}
\]
\[
\sigma_H(\gamma; z) = \sigma (1 + \gamma) (1 - z/H)^{\gamma},
\]

(44)

with \( \gamma \geq 0 \) (in this case, 2/3) and \( \sigma \) being the mean
extinction (obtained, say, from cloud optical depth \( \tau = \sigma H \)).

[55] This stratification in \( \sigma \) will directly affect the spatial
(9) and temporal (8) observables in MSL, even if it does not
affect the cloud’s albedo in (7). Indeed, the local value of
the MFP will be different at the top and bottom of the cloud
and, physically, this means that the random walk representing
the diffusing light propagation is scaled up (near cloud base)
or down (near cloud top). Since MSL systems have
already probed clouds from both sides, and will continue to
do so, it is imperative to quantify the effect of stratification
on the observables.

[56] Now, because \( g \) is assumed constant, \( \sigma(z) \) will
have the same behavior as \( \sigma_{g,H}(\gamma; z) \). However, the vanishing
\( \sigma(z) \) at either \( z = 0 \) (lidar below cloud) or \( z = H \) (lidar above
cloud) is problematic for the diffusion model. Indeed, the
BCs in (27) make necessary the evaluation of \( F_c(t, \tilde{r}, z) \) in
(20) for \( z = 0 \) and \( z = H \), one of which contains a division by
\( \sigma(z) = 0 \). Physically, the local transport MFP is divergent and
diffusion, as an approximation to RT, fails near one of
the cloud boundaries (symptomatically, the associated ex-
trapolation length is infinite).

[57] Instead of the troublesome power law model, we can take

\[
\sigma_{\ell}(z) = \sigma \times [1 + \Delta (z/H - 1/2)]
\]

(45)

where \( |\Delta| < 2 \) is the relative difference in extinction at
the two cloud boundaries with respect to the mean, and
similarly for \( \sigma_{H}(z) \) using \( \sigma_{H} = (1 - g) \sigma \). The least-squares
difference between the linear model in (45) and a given
power law in (44) is minimized by the choice

\[
\Delta(\gamma) = \pm 6 \times \left( \frac{2 \gamma + 1}{\gamma + 2} - 1 \right)
\]

(46)

where + is mapped to \( \sigma_{\ell}(\gamma; z) \) and − to \( \sigma_{H}(\gamma; z) \). Values of
special interest are \( \Delta = \pm 3/2 \) since they approximate \( \gamma = 2/3 \),
the above-mentioned expectations based on parcel theory for
a cloud illuminated from below and above respectively.

[58] Ideally, we would like to extend the new collimated/
anisotropic internal source model to the case of internal
variability. However, the resulting ODE problem does not
appear to be analytically tractable. We therefore revert to
the isotropic boundary point source model used by Davis et al.
[1999]. We thus wish to solve both the space domain \( (s = 0) \)
problem of MSL,

\[
\bar{F}^s = -\left[ k^2/3\sigma_{s,H}(z) \right] \bar{J}, \bar{J}' = -3\sigma_{s,H}(z) \bar{F},
\]

(47)

and its time domain \( (k = 0) \) counterpart,

\[
\bar{F}^t = -(s/c)\bar{J}, \bar{J}' = -3\sigma_{s,H}(z) \bar{F},
\]

(48)

in both cases, subject to

\[
\bar{J} - 3\chi\bar{F}^t_{2|z=0} = 4, \bar{J} + 3\chi\bar{F}^t_{2|z=H} = 0,
\]

(49)

leaving \( \chi \) as an unspecified parameter. By inspection, we see
that nondimensional cloud responses can only depend on
\( \Delta \) and \( \tau = \sigma_{H} \). In contrast with the \( \Delta = 0 \)
[Davis et al., 1999; Love et al., 2001a], we will not have
similar behavior between \( s/c \) and \( k^2/3\sigma_{H} \), since they are
interchangeable in \( \sigma_{s,H}(s, k) \) only when \( \Delta = 0 \).

[59] Spatial and temporal moments are computed as
previously: (1) solve boundary value problem for coupled
ODEs; (2) obtain \( R(s, 0) \) from \( J(s, 0; z) = 0 \), or \( R(0, k) \) from
\( J(0, k; z) = 0 \); (3) expand into a Taylor series of the desired
length in the variable of interest and extract the moments of
interest; and (4) translate result into a high-level program-
ing language for easy manipulation and plotting. A
computer-assisted algebra tool is highly recommended for
all of the above steps.

[60] As it turns out, the Fourier domain (spatial) diffusion
problem in (47) and (49) for \( J(0, k; z) \) is solvable in terms of
order-zero and order-one modified Bessel functions of the
first and second kinds. The Laplace domain (temporal)
diffusion problem in (48) and (49) for \( J(s, 0; z) \) is solvable
in terms of Airy functions and their derivatives, which are
related to modified Bessel functions with 1/3-integer orders.
At zeroth order, we retrieve (using L’Hôpital’s rule)

\[
R = 1 - T, \quad T = \frac{1}{1 + \tau/2\chi},
\]

(50)

the well-known [Schuster, 1905; Meador and Weaver,
1980] expression for cloud transmittance \( T \) for diffuse
illumination in the absence of absorption, and cloud albedo
\( R \). As expected, they are not sensitive to internal structure
since optical properties \( \sigma_0 \) and \( g \) are held constant. For
MSL observables proper, higher-order terms in \( k \) and \( s \) yield
Overall, the diffusion results are just slightly offset from their MC counterparts. The most remarkable difference between diffusion and MC is the logarithmic divergence of \( \langle \rho^2 \rangle \) at \( \Delta = 0^+ \) that is manifest in (51). All is as if the effective diffusivity constant \( \langle \rho^2 \rangle / \ell_q \), as observed by MSL at the cloud boundary, becomes infinite with the value of the transport MFP at \( z = 0 \), namely, \( \ell_q(0) = 1/\sigma(1 - \Delta/2) \).

As stated above, diverging \( \ell_q \) is clearly a challenge for the diffusion model since trajectories become ballistic, and apparently more so when this occurs near the source. This problem could probably be fixed by introducing a parameterization \( \chi(\Delta) \) where \( \chi = 0 \) as \( \Delta = 2\pi \).

We also note in the log-lin plot that the MC results for \( \langle ct \rangle \) and \( \langle \rho^2 \rangle / \ell_q \) are quasi-linear in \( \Delta \) over its full range. This is especially good for spatially resolved MSL observations from ground (\( \Delta > 0 \)) since, instead of the diffusion model per se, one can use a log-linear extrapolation from the log sensitivity of these moments to \( \Delta \) evaluated at \( \Delta = 0 \).

Figure 4 illustrates the outcome of the above diffusion model for any moment value at \( \Delta = 0 \) from a more accurate model (e.g., from the previous section), or even a tabulated MC result; that hybrid approach should further reduce any bias.

### 5.2. Random Variability

Barker and Davis [2005] showed that there are two broad classes of models in 3-D RT that go after the large-scale effects of unresolved small-scale variability in cloud structure, which is invariably assumed random. Members of one class of mean field theory end with new transport equations to solve. Members of the other class pursue homogenization: redefine coefficients in 1-D RT so that the known solutions of that problem capture the main 3-D effects, which is clearly the path of least resistance. Among these “effective medium” approaches to random 3-D variability, we favor the rescaling technique by Cairns et al. [2000]. Although it is a one-parameter solution, it stems from a careful renormalization treatment of both one- and two-point statistics, i.e., the PDF of \( \sigma(\tau) \) and its autocorrelation function respectively.

Starting with the \( \delta \)-rescaled (primed) quantities in (42) that account for problematic forward scattering peaks, we have:

\[
\sigma''(z) = (1 - \epsilon)\sigma'(z), \quad (1 - g') = (1 - g)[1 - \epsilon/(1 - \epsilon)],
\]

in the case of conservative scattering, where \( \epsilon \) is the variability parameter. While \( \delta \) rescaling leaves the product \((1 - g)\sigma\) invariant, it decreases here both through \( \sigma \) and through \( 1 - g \) as \( \epsilon \) increases (since \( g'' > g' \)).

Letting overscores denote averages over the spatial disorder, Cairns et al. show specifically that for moderate-amplitude 3-D effects one has

\[
\epsilon = a - \sqrt{a^2 - V}
\]
Table 1. Properties of a Cloud With \( H = 0.5 \) km Probed by MSL From Both Sides in Table 2

<table>
<thead>
<tr>
<th>Parameter of interest</th>
<th>Symbol</th>
<th>Unit</th>
<th>Given</th>
<th>( \delta )-Scaled</th>
<th>Caims-3D</th>
<th>Above</th>
<th>Below</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter value</td>
<td></td>
<td></td>
<td>0</td>
<td>0.72</td>
<td>0.15</td>
<td>-1.5</td>
<td>+1.5</td>
</tr>
<tr>
<td><strong>Cloud Optical Properties</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Optical depth</td>
<td>( \tau )</td>
<td>[-]</td>
<td>0.67</td>
<td>0.67</td>
<td>0.58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Asymmetry factor</td>
<td>( g )</td>
<td>[-]</td>
<td>1.04</td>
<td>0.91</td>
<td>0.99</td>
<td>0.74</td>
<td>1.32</td>
</tr>
<tr>
<td>Scaled optical depth</td>
<td>( (1 - g)\tau )</td>
<td>[-]</td>
<td>0.30</td>
<td>0.28</td>
<td>0.34</td>
<td>0.26</td>
<td>0.46</td>
</tr>
<tr>
<td><strong>Cloud Radiative Responses</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Albedo</td>
<td>( R )</td>
<td>[-]</td>
<td>0.22</td>
<td>0.29</td>
<td>0.15</td>
<td>0.36</td>
<td>0.063</td>
</tr>
<tr>
<td>Mean in-cloud path</td>
<td>( \sqrt{(\rho^2)} )</td>
<td>km</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMS lateral transfer</td>
<td></td>
<td></td>
<td>0.063</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average MSL radiance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where \( V = \pi^2/\sigma^2 - 1 \) (variance relative to mean-squared) and \( 2a = 1 + 1/\sigma l_c \). We denote here the characteristic correlation scale of the spatial variability by \( l_c \). We see that small-scale fluctuations, i.e., when \( l_c \ll \text{MFP} \approx 1/\tau \), have little effect since \( \epsilon \approx 2V/(2a) \ll 1 \) (irrespective of \( V \) as \( a \) becomes very large. Large-scale fluctuations (\( \sigma l_c \gtrsim 1 \)) can have a strong impact (\( \epsilon \lesssim 1 \)); however, this stretches the validity of the model (in particular, amplitude is then limited to cases where \( V \lesssim a^2 \lesssim 1 \)). For very large-scale fluctuations (\( \sigma l_c \gg 1 \), hence \( a \approx 1/2 \), hence \( V \lesssim 1/4 \)), it is clear that one should average over macroscopic MSL responses rather than try to find an effective medium to account for variability effects in the spirit of an Independent Pixel Approximation [e.g.] and references therein Barker and Davis, 2005]. That is precisely how Davis and Marshak [2002] approached the problem of spatial variability: following Barker [1996], they averaged expressions similar to those in (41), but for transmission, over a Gamma distribution of \( \sigma \) values.

[67] The above scale-by-scale breakdown of spatial variability impacts is consistent with the first-principles analysis by Davis and Marshak [2004] who, incidentally, show that the actual MFP is \( 1/\sigma \) in a broad class in variable media, and this always exceeds \( 1/\sigma \) (they are equal only when \( \sigma \) is uniform).

**6. Signals and Noises in Multiple-Scattering Cloud Lidar**

**6.1. Signal-to-Noise Ratio Estimation**

[68] Before we consider the use of MSL data for cloud remote sensing purposes, it is standard procedure to evaluate the typical signal as realistically possible given the specifications of actual or proposed instruments. At the same time it is important to quantify all foreseeable sources of noise and, from there, estimate a priori the signal-to-noise ratio (SNR). The main purposes of such exercises are to test ideas in instrument design as well as to experiment with different sampling strategies for observations, in this case, of different types of cloud.

[69] We now illustrate this key modeling application of the above theoretical results for MSL observables. In this demonstration, we will focus on two specific MSL systems that probe clouds from either side:

[70] 1. The Wide-Angle Imaging Lidar (WAIL) is a ground-based design developed at Los Alamos National Laboratory. It has already been deployed several times in New Mexico [Love et al., 2001a, 2001b] and once in Oklahoma [Polonsky et al., 2005]. Yet the current engineering model is still being refined and we will use here parameters for so-far untested hardware.

[71] 2. The Lidar-in-space Technology Experiment (LITE) was the first demonstration of space-based lidar conducted, largely from NASA’s Langley Research Center, as a payload in the cargo bay of Space Shuttle Discovery during flight STS-64 [Winker et al., 1996]. This mission was flown 9–20 September 1994, and was considered a vast success that indeed blazed the path for current and future lidar satellite missions, including NASA’s ICESat/GLAS and CALIPSO/CALIOP.

[72] We use the same cloud in both cases: a typical boundary layer stratus deck at 0.7 km altitude, with \( H = 0.5 \) km, \( \tau = 25 \), and \( g = 0.85 \). Table 1 summarizes all the intermediate time-dependent 3-D RT modeling results leading to the temporally, spatially and angularly averaged radiance excited by the pulsed laser illumination that escapes the illuminated/observed cloud boundary. We estimate this basic quantity from

\[
I_{\text{obs}} \approx \frac{R}{\pi} \frac{\pi}{\langle \rho^2 \rangle} (t),
\]

and it will be expressed in its natural radiometric units, namely, photons per laser photon, per unit of aperture area, per steradian of FOV, and per unit of exposure time. At sufficient accuracy, this is also the radiance detected across (essentially empty) space by the MSL receiver.

[73] Standoff distances \( d_{\text{obs}} \) are of course very different: 0.7 km for WAIL, ≈259 km for LITE. The next question is what solid angle and viewing angle are subtended by the \( \approx \pi \langle \rho^2 \rangle \) circular area of the observed radiative Green function. This “adapted” solid angle is given by

\[
\delta \Omega_{\text{obs}}(d_{\text{obs}}) = \frac{2 \pi}{d_{\text{obs}}^2 \langle \rho^2 \rangle + 1} = 2 \pi (1 - \cos \theta_{\text{Gf}}).
\]

The corresponding adapted viewing angle \( \theta_{\text{Gf}}(d_{\text{obs}}) \) is 45.5° for WAIL and 0.06° (≈1 mrad) for LITE. Ideally, one wants the MSL instrument’s FOV (≈2 × \( \theta_{\text{max}} \)) to be at least this large, and not too much more. Say, \( \tan \theta_{\text{max}} \approx 2 \) to 3 times
Table 2. Specifications for Two MSL Systems and Expected SNRs

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Unit</th>
<th>LITE</th>
<th>WAIL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standoff distance</td>
<td>(<em>d</em>{ob} )</td>
<td>km</td>
<td>259</td>
<td>0.7</td>
</tr>
<tr>
<td>Equation (56)</td>
<td>(\delta \Omega_{\text{FOV}} )</td>
<td>sr</td>
<td>4 (10^{-6} )</td>
<td>1.8</td>
</tr>
<tr>
<td>“Adapted” FOV*</td>
<td>2(\delta \Omega_{\text{FOV}} )</td>
<td>deg</td>
<td>0.06</td>
<td>91</td>
</tr>
</tbody>
</table>

**Transmitter Parameters**

- **Wavelength**\( \lambda \) nm 532  532  
- **Power** \(E_p \times P \) W 5  5
- **RepRate** Hz 10  12  10\(^3\)
- **Pulse energy** \(E_p \) mJ 500  0.42
- **Photons per pulse** \(E_p / hc \) [-] \(1.34 \times 10^{18} \) \(1.12 \times 10^{15}\)

**Receiver Parameters**

- **Optical throughput** \(OTp \) % 45\(^b\) | 70 |
- **Quantum efficiency** \(\eta_{\lambda} \) % 14  70
- **Aperture area** \(A \) m\(^2\) 0.63  \(10^{-5}\)
- **(effective) FOV** \(2\delta \) deg 0.20  88
- **\(\delta \Omega_{\text{FOV/1UR_to_F}} \) (\(1 - \cos \theta_{\max} \)) /2 \([-\] | 1 + 3 \(10^{-6}\) | 1.163
- **Etendue** \(A \times \text{1UR} \) \(m^2 \) sr 6 \(10^{-6}\) \(15.5 \times 10^{-6}\)
- **Filter bandpass** \(\Delta \lambda \) nm 0.35  50

**Sampling and Averaging**

- **Path bin size** \(\delta t \) m 10  10
- **Integration time** \(\Delta t \) s 0.1  300
- **Number of pulses** \(N_p \) [-] 1  3.6 \(10^6\)

**Predictions**

- **Signal** \(S_{\text{sig}}(\Delta t) \) counts 2721\(^b\)  341 \(10^3\)
- **Only shot noise** \(\text{SNR}_{\text{shot}} \) [-] 78  584
- **+ lunar background** \(\text{SNR}_{\text{lunar}} \) [-] 38  19
- **+ solar background** \(\text{SNR}_{\text{sun}} \) [-] 4.3 \(10^{-2}\)  2.6 \(10^{-2}\)

\(^{a}\)This happens to be slightly more (WAIL) or, as recommended in the main text, somewhat smaller (LITE) than the actual FOV \(2\delta \Omega_{\text{FOV}}\).

\(^{b}\)Nighttime value for LITE; daytime value is 20%, which affects equally signal and background, hence an SNR decrease by \(\sqrt{20/45} \approx 0.67\) accounted for in the tabulated value in the last row.

\(^{c}\)As explained in text, this pulse budget is distributed equally among path bins; assuming 272 bins, \(N_p = 3.6 \times 10^5/272 \approx 13.2 \times 10^3\) (an integration over 1.1 s per bin).

\(\langle \rho \rangle^{1/2}/d_{ob}\). Indeed, the observed radiance field is concentrated in this region of direction space. As it turns out, LITE satisfies this constraint and WAIL almost does. If that were not the case, \(\theta_{\max}\) too small means loss of signal while too big adds little signal in view of its exponential decay in space (only more background noise is collected).

\([74]\) Table 2 lists the relevant transmitter and receiver parameters for LITE (as flown) and WAIL (new/untested configuration). To illustrate SNR estimation we focus on the temporal aspect of MSL, which is the only one LITE could access in detail. We therefore seek an expression for the number of “photon counts” \(S_{\text{sig}}(\Delta t)\) registered in a typical path length bin of size \(\delta t\) when integrating MSL signal over all of space but only a given time interval \(\Delta t\). For the average time-dependent radiance in (55), basic radiometry gives us

\[
S_{\text{sig}}(\Delta t) \approx OTp \times \eta_{\lambda} \\
\times \left( \frac{E_p}{hc} \times I_{\text{obs}} \right) \\
\times \left( A \times \text{1UR} \right) \times \delta t \\
\times \left( \text{RepRate} \times \Delta t \right) \\
\times \text{BG}(\Delta t) \approx OTp \times \eta_{\lambda} \\
\times \left( \frac{\mu_0 F_\alpha}{hc} \right) \\
\times \left( A \times \text{1UR} \right) \times \delta t \\
\times \text{BG}(\Delta t) \times \text{electronics} \\
(57)
\]

where \(\Delta \lambda\) is the width of the background suppression filter (see Table 2), \(\mu_0\) is the cosine of the solar/lunar zenith angle, \(T_{\text{dif}}(\mu_0)\) is cloud’s diffuse transmittivity to the ground (but we use the cloud’s reflectivity \(R(\mu_0)\) if the MSL is above it), and \(F_\alpha\) is the solar/lunar spectral flux incident on the top of the atmosphere. For the solar background, we have \(F_\alpha = 1.869 \text{ W/m}^2/\text{nm}\) at \(\lambda = 532 \text{ nm}\) [American Society for Testing and Materials, 2000]. For the nighttime counterpart, the Moon is assumed to be a Lambertian reflector (albedo 0.12) subtending \(35^\circ\) of arc (solid angle modulated by phase angle) receiving the same irradiation.

\([76]\) Being three independent sources of noise, their variances are additive, and total RMS noise amplitude is

\[
N_{\text{elec}}(\Delta t) \approx \sqrt{S_{\text{elec}}(\Delta t) + \text{BG}(\Delta t) + \text{electronics}} \\
(59)
\]
We will assume that there are just enough bins, each $\delta t$ wide, to cover $3 \times r'$ where

$$t' = (H/c) \times \frac{(1 - g)r}{3(\pi R)^2}, \quad (60)$$

is the exponential decay rate of the Green function in time [Polonsky and Davis, 2004, or Part 2], with $R$ being obtained from $(1 - g)r$ in (50) with $\chi = 2/3$. For the cloud of present interest, we find $ct \approx 0.85$ km in units of path. This rationale leads to 272 time bins, and $\Delta t$ is set to 5 min (300 s, $3.6 \times 10^6$ pulses), which is basically enough for ~1.5 km of cloud to advect by at nominal wind speeds (~5 m/s). This is not far from optimal since any faster sampling of the radiative Green function would have overlap due to the horizontal transport. In contrast, LITE is moving along its orbit at 7 km/s. In this case, we want to get signal from every pulse, hence $\Delta t = 1$/RepRate = 0.1 s.

6.2. Discussion

[76] The last three rows of Table 2 give

$$\text{SNR} = S_{\text{rel}}(\Delta t) / N_{\text{rel}}(\Delta t) \quad (61)$$

for the two MSL systems (neglecting electronic noise) under three scenarios: moonless night, moon present, and daytime.

[79] In the last two situations, we need to compute $F = T_{\text{dif}} R$, for which it suffices to use predictions from two-stream theory [Meador and Weaver, 1980] for the given cloud (after $\delta$-Eddington and Cairns rescalings). We take $\mu_0 = 0.5$ for WAIL and cos $51^\circ$ for LITE. This last value is for the actual zenith angle of the moon at 6:53 GMT on 9/16/94 viewed from 36°N by 128.6°W, which is approximately when and where the LITE data analyzed in Part 2 was captured during nighttime orbit # 135 (the moon was 87% full). For WAIL, we assume an average solar/lunar zenith angle (and, worst case scenario, a full lunar disk).

[80] We conclude from these SNR estimates that the two radically different MSL systems have and will perform well at night, even with a full moon. However, daytime operation remains a challenge: the SNR must be boosted by at least $\sim 10^3$, and increasing $\Delta t$ by $10^6$ is clearly not an option. Current plans at LANL for the WAIL project [Love et al., 2001b] involve an ultranarrow $(\Delta \lambda \approx 5$ pm) magneto-optic (a.k.a. Faraday) filter centered on one of the sodium lines in its strong doublet near 589 nm, where the sun $(F_{\text{sun}})$ is already about $20 \times$ dimmer than at nearby wavelengths. Contrary to interference-based monochromators, such filters have a very wide acceptance angle, which is critical to the present application. Even factoring in that $2F_{\text{sun}}$ is somewhat larger at 589 nm than at 532 nm, we are close to our goal. However, a sufficiently stable and powerful tunable laser is required to utilize this sophisticated background rejection technique.

[81] Another approach altogether is to complement operational MSL systems with high-resolution oxygen A-band spectrometers, a passive technique in the solar spectrum. The main products of these instruments for cloudy skies are indeed the successive moments of path length [Pfeilsticker et al., 1998; Min and Harrison, 1999], i.e., $(\langle c r \rangle^q)$, $q = 1, 2, 3$ (maybe more). All time-only cloud MSL remote sensing techniques are therefore amenable to these data. It is important to keep this in mind when NASA launches in late 2008 the first high-resolution oxygen A-band spectrometer into space on the Orbiting Carbon Observatory (OCO) mission [Crisp et al., 2004].

7. Summary

[82] We have considerably refined the diffusion-theoretic forward model for predicting space-time signals from multiple-scattering cloud lidar. More precisely, we targeted moments of the observable space-time Green function as expressions of the cloud’s physical thickness $H$ and optical depth $\tau$ (considered as the remote sensing unknowns), and the asymmetry factor $g$ of the scattering phase function. In the original model by [Davis et al., 1999], moments (normalized by $H$) were only functions of scaled optical depth $(1 - g)r$. We have added to that capability (1) accurate representation of the pulsed laser source as a collimated anisotropic exponentially decaying spatial distribution of internal sources, and consequently separation of the (smooth) scattered and (singular) uncollided components of radiance; (2) $\delta$-Eddington rescaling (that preserves $(1 - g)r$) further improves the above refinement by partially accounting for the strong forward scattering peak in the phase functions of observed cloud droplet populations (since normalized moments are now functions of $\tau$ and $g$); (3) parameterization of the impacts of internal cloud stratification using an analytically tractable model to compute the sensitivity of moments to a constant gradient in extinction $\sigma(z) \propto 1 + \Delta z \times (\sigma/H - 1/2)$; (4) use of the above linear gradient model to mimic the more relevant case of power law behavior, as an important instance, extinction increasing as a $2/3$ power from cloud base is mapped to $\Delta = 2[\sigma(H) - \sigma(0)]/[\sigma(H) + \sigma(0)] = \pm 3/2$; and (5) Cairns et al. [2000] rescaling that changes $\tau$ and $g$ (without conserving $(1 - g)r$), which defines an effective optical medium that captures the systematic effects on space-time cloud responses of turbulence-driven random internal variability at scales up to a few mean free paths. All but the last item received at least limited validation by comparison with Monte Carlo simulations. Items 3–4 are critically important as one switches between illumination/observation of the cloud from below (ground-based probes) and from above (airborne or space-based systems).

[83] This effort brings the diffusion modeling project to a state of balance between formal sophistication and practical utility. The main drivers for this development are (1) physical insights gained from a PDE-based approach, (2) accuracy sufficient for applications in instrument and/or algorithm design, (3) flexibility in the representation of cloud structure as well as of radiation sources and sinks, and (4) extreme computational efficiency of analytical methods that enables real-time data processing when and where multiple-scattering cloud lidars will be deployed operationally on ground or in space.

[84] As an example of activity in algorithm design, the refined model’s features are showcased with a demonstration of how $H$ and $\tau$, hence extinction $\sigma = \tau/H$, can be derived from multiple-scattering cloud lidar data of sufficient quantity and quality to estimate selected moments.
respectively. The retrieved cloud properties are representative of large-volume averages, on the order of \(H^3(1-g)r\).

As an example of activity in instrument design, the improved diffusion model is applied to signal-to-noise ratio (SNR) estimations for two very different multiple-scattering lidar systems. SNR is computed a priori for a new, so far untested, configuration of LANL’s Wide-Angle Imaging Lidar (WAIL); it is also computed a posteriori for NASA’s Lidar-In-space Technology Experiment (LITE). Part 2 of this series (Davis et al., manuscript in preparation, 2008) will demonstrate innovative ways of extracting cloud properties from real data collected with these two systems, and compare their outcome with available cloud information from other sources.

If even more realistic representations of the scattering phase function and/or 3-D cloud structure are needed in the applications, without paying too high a price in CPU cycles, then one should turn to ultraefficient numerical techniques. For instance, data exploitation in the two airborne multiple-scattering cloud lidar systems in existence use (1) a neural network trained with one-time 3-D Monte Carlo runs [Evans et al., 2003, 2006] and (2) a multidimensional look-up-table populated with one-time 1-D Monte Carlo runs [Cahalan et al., 2005]. Another promising approach, which is more closely related to the present diffusion model, would use the rapid numerical time-dependent two-stream solver developed recently by Hogan and Battaglia [2008]. In particular, its representation of the pulsed laser source uses a small-angle multiple forward scattering model by Hogan [2008] that is of practical interest in its own right for processing (on- or near-beam) lidar data from optically thin clouds and optically thick aerosol layers. Although designed for probing dense clouds, deployed multiple-scattering lidar systems will continue to collect data under such semiclear skies. It will be interesting to see what added value they can contribute to aerosol and cirrus studies.

Acknowledgments. The authors are grateful for financial support from the U.S. DOE Atmospheric Radiation Measurement (ARM), project SCFY041020, and from LANL’s Laboratory-Directed R&D (LDRD) programs. We thank Luc Bissonnette, Bob Cahalan, Frank Evans, Robin Hogan, Steve Love, Matt McGill, Alexander Marshak, Igor Polonsky, James Spinhirne, Tamas Varos, James Bretherton, and K. Yetzer (2005), THOR, cloud TThickness from Offbeam lidar Returns, J. Atmos. Oceanic Technol., 22, 605–627.

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